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An Extended Galilei Group and its Application

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Abstract

The Galilei group is combined with two one-dimensional groups, to form a twelve-dimensional extended Galilei group. Irreducible representations of this group depend upon two indices m, s that can, respectively, be interpreted as the mass and spin of a non-relativistic particle. It turns out that the true irreducible representations of the ordinary Galilei group correspond to $m = 0$, and this explains why these representations have no physical interpretation.

1. *Introduction*

Inönü & Wigner (1952) have shown that none of the true irreducible representations of the Galilei group have a physical interpretation. Only the ray representations of the Galilei group (Bargmann, 1954) have a physical interpretation. The result obtained by Inönü and Wigner is surprising, because the Galilei group is of fundamental importance in non-relativistic quantum mechanics. However, non-relativistic quantum mechanics is covariant under a larger group of transformations than the Galilei group. Consider the transformations

$$\psi \to \exp(i\alpha t)\psi \tag{1.1}$$

$$\psi \to \exp(i\beta)\psi \tag{1.2}$$

where α and β are real numbers and t is the time. The transformation (1.1) represents a change in the normalisation of the potential energy, (1.2) is a simple change of phase. Strictly speaking (1.1) and (1.2) are unitary transformations, under which the state of a non-relativistic particle is invariant. In the present work these transformations are combined with the Galilei group to form a larger group that we call the extended Galilei group. The Lie algebra of this group has irreducible representations depending upon two indices that can be interpreted as the mass and spin of a particle.

2. *The Extended Galilei Group*

The representatives of the infinitesimal generators, and the parameters of the extended Galilei group are given in the following table:

	Transformation	Generator	Parameter
I	Pure Galilei	$\mathbf{G} = (G_1 G_2 G_3)$	$\mathbf{v} = (v_1 v_2 v_3)$
II	Spatial translation	$\mathbf{T} = (T_1 T_2 T_3)$	$\mathbf{a} = (a_1 a_2 a_3)$
III	Time translation	D	τ
IV	Rotation	$\mathbf{J} = (J_1 J_2 J_3)$	$\mathbf{u} = (u_1 u_2 u_3)$
V	Change of energy normalisation	G_4	γ
VI	Change of phase	C	κ

By the extended Galilei algebra we understand the relations

$$\begin{aligned} &[G_i, G_k] = 0, \quad [T_i, T_k] = 0, \quad [\mathbf{T}, D] = 0 \\ &[T_i, G_k] = C\delta_{ik}, \quad [\mathbf{G}, D] = \mathbf{T} \quad (i, k = 1, 2, 3) \end{aligned} \tag{2.1}$$

$$[G_4, D] = C, \quad [G_4, \mathbf{G}] = 0, \quad [G_4, \mathbf{T}] = 0 \tag{2.2}$$

$$[C, \mathbf{G}] = 0, \quad [C, G_4] = 0, \quad [C, \mathbf{T}] = 0, \quad [C, \mathbf{D}] = 0 \tag{2.3}$$

together with the relations involving **J**, that are the same as in the ordinary Galilei algebra. C and G_4 are scalars. C commutes with all infinitesimal generators, and is an invariant of the group. In every irreducible representation of the algebra, we have

$$C = imI \tag{2.4}$$

where I is the unit operator and m is a number. If $m = 0$, the extended algebra reduces to the ordinary Galilei algebra. Assuming $C \neq 0$, we define

$$T_4 = D + \tfrac{1}{2}C^{-1}\mathbf{T}^2 \tag{2.5}$$

T_4 commutes with all generators except G_4, and we may write

$$[T_\mu, T_\nu] = 0, \quad [G_\mu, G_\nu] = 0, \quad [T_\mu, G_\nu] = C\delta_{\mu\nu} \quad (\mu, \nu = 1, 2, 3, 4) \tag{2.6}$$

This is the familiar Heisenberg algebra, if C is pure imaginary. For each real value of m there exists only one irreducible representation $H9(m)$ of this algebra (von Neumann, 1931).

It is from now on assumed that T_μ, G_ν are infinite dimensional anti-hermitean matrices. The matrices

$$\mathbf{G}, \mathbf{T}, D = T_4 - \tfrac{1}{2}C^{-1}\mathbf{T}^2, \qquad \mathbf{J} = C^{-1}\mathbf{T}\times\mathbf{G}, G_4 \tag{2.7}$$

satisfy all the relations of the extended Galilei algebra, and $H9(m)$ is thus an irreducible representation of the extended Galilei group. The Lie algebra of the rotation group is a subalgebra of the Galilei algebra, and denoting by $R(s)$ an irreducible representation of the rotation group, an irreducible representation of the extended Galilei group is given by

$$G(m,s) = H9(m) \otimes R(s)$$

$$-\infty < m < +\infty, \qquad m \neq 0, \qquad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

The hermitean matrix $-iT_4$ commutes with the generators $\mathbf{G}$, $\mathbf{T}$, D, $\mathbf{J}$ of the Galilei group, and is a Galilei invariant. We denote by $|\alpha,m,s\rangle$ an eigenvector of $-iT_4$, where

$$(iT_4 + \alpha)\,|\alpha, m, s\rangle = 0 \tag{2.8}$$

The linearly independent vectors that satisfy (2.8) span a subspace of $G(m,s)$ that we denote by $G(\alpha,m,s)$, and this subspace is irreducible with respect to Galilei transformations. We have thus obtained an irreducible representation of the Galilei group [ray representation because of the algebra (2.1)] that depends upon three indices α, m, s. The sum of all Galilei invariant subspaces $G(\alpha,m,s)$, for all values of α between $-\infty$ and $+\infty$, form the representation space $G(m,s)$ of the extended Galilei algebra.

3. *Application to Quantum Mechanics*

Define the unitary matrices

$$\begin{aligned} U(\mathbf{v}) &= \exp(\mathbf{v}.\mathbf{G}), \qquad & U(\mathbf{a}) &= \exp(-\mathbf{a}.\mathbf{T}), \\ U(\tau) &= \exp(-\tau D) \qquad & U(\mathbf{u}) &= \exp(\mathbf{u}.\mathbf{J}), \end{aligned} \tag{3.1}$$

$$U(\gamma, \kappa) = \exp(\gamma G_4 + \kappa C) \tag{3.2}$$

A Galilei transformation is represented by a matrix $U(\mathbf{a},\tau,\mathbf{v},\mathbf{u})$ that is a product of the matrices (3.1), where the positions of the factors $U(\mathbf{u})$ and $U(\tau)$ are, respectively, to the right and to the left of $U(\mathbf{v})$. The position of the factor $U(\mathbf{a})$ is arbitrary, except for being to the left of $U(\mathbf{u})$. An element of the extended Galilei group is represented by the matrix

$$U = U(\gamma, \kappa)\, U(\mathbf{a}, \tau, \mathbf{v}, \mathbf{u}) \tag{3.3}$$

By the help of the unitary matrix

$$V = \exp(-\tfrac{1}{2}\mathbf{T}^2\, G_4\, C^{-2})$$

we define the hermitean matrices

$$X_\mu = C^{-1}\, V^+ G_\mu\, V, \qquad P_\mu = -iV^+ T_\mu\, V \tag{3.4}$$

where

$$[X_\mu, X_\nu] = 0, \qquad [P_\mu, P_\nu] = 0, \qquad [X_\mu, P_\nu] = i\delta_{\mu\nu}$$

By making use of the familiar calculation rules of the Heisenberg algebra, we obtain the transformation equations

$$U^{+}\mathbf{X}U = R\mathbf{X} + \mathbf{v}X_4, \qquad U^{+}X_4U = X_4 + \tau \tag{3.5}$$

$$\begin{gathered} U^{+}\mathbf{P}U = R\mathbf{P} + m\mathbf{v}, \qquad U^{+}P_4U = P_4 - \mathbf{v}.R\mathbf{P} - \tfrac{1}{2}m\mathbf{v}^2 + m\gamma \\ R\mathbf{X} = U^{+}(\mathbf{u})\mathbf{X}U(\mathbf{u}), \qquad R\mathbf{P} = U^{+}(\mathbf{u})\mathbf{P}U(\mathbf{u}) \end{gathered} \tag{3.6}$$

These equations make it reasonable to interpret $\mathbf{X}$ as the position operator, $\mathbf{P}$ as the momentum operator, and X_4 as the time operator of a particle with mass $m = -iC$. With $\gamma = 0$, the operator

$$E = -P_4 = iD$$

transforms in the same way as the kinetic energy of the particle. E is therefore interpreted as the energy operator.

Under the transformation U of the full group, a wave function $\psi(x) = \langle x|\psi\rangle$ transforms into

$$U\psi(x) = \langle x|U|\psi\rangle = \exp[i(\alpha t + \beta)]\,\psi(\Gamma^{-1}x) \tag{3.7}$$

where $\langle x|$ is a simultanous eigenbra of X_μ,

$$\psi(\Gamma^{-1}x) = \langle\Gamma^{-1}x|\psi\rangle = \langle x|U(\mathbf{a}, \tau, \mathbf{v}, \mathbf{u})|\psi\rangle,$$

$$\langle x|U(\gamma, \kappa) = \exp[i(\alpha t + \beta)]\langle x|$$

$\Gamma^{-1}x$ denotes the inverse transformation of (3.5), α and β are real numbers. The condition

$$|U\psi(x)|^2 = |\psi(\Gamma^{-1}x)|^2 \tag{3.8}$$

is thus satisfied under the extended group also.

A particle state is characterised by the mass m, the spin s, and the normalisation constant α of the energy. A particle is therefore represented by a Galilei invariant subspace $G(\alpha,m,s)$. However, in contrast to m and s, the value of α is arbitrary. Although the normalisation of the energy is arbitrary, it must be fixed, in order that the particle shall have a well-defined energy. The transformation $U(\gamma)$ changes the particle representation from the subspace $G(\alpha,m,s)$ to another physically equivalent subspace $G(\alpha',m,s)$. Now

$$-iT_4 = P_4 + \frac{\mathbf{P}^2}{2m}$$

and from (2.8) we get the Schrödinger equation

$$\langle x|\left(P_4 + \frac{\mathbf{P}^2}{2m} - \alpha\right)|\alpha, m, s\rangle = 0$$

The conventional Schrödinger state space is given by

$$G(0, m, s)$$

Within the subspace $G(\alpha, m, s)$ we define the scalar product

$$(\phi, \psi) = \int \langle \phi | x \rangle \langle x | \psi \rangle \, d^3 x$$

where the physically realisable states ϕ and ψ satisfy the relations

$$(\phi, \psi) = (U\phi, U\psi), \qquad (\phi, X_\mu \psi) = (X_\mu \phi, \psi), \qquad (\phi, P_\mu \psi) = (P_\mu \phi, \psi)$$

4. *Conclusion*

We have shown that a particle with mass m and spin s is represented by a Galilei invariant subspace $G(\alpha, m, s)$ of the representation space $G(m, s)$ of the extended Galilei group. All subspaces $G(\alpha, m, s)$ are physically equivalent representatives of the particle, and this necessitates transformations from one subspace to another. For this reason an extension of the Galilei group is introduced. For $m = 0$, the extended Galilei algebra reduces to the ordinary Galilei algebra. The representations obtained by Inönü and Wigner should therefore correspond to non-relativistic particles with zero mass, and this explains why a physical interpretation of these representations is impossible.

All the dynamical variables, including the mass and the time, are functions of the infinitesimal generators of the extended Galilei group. In this respect the present theory has an advantage in comparison with the standard theory.

References

Bargmann, V. (1954). *Annals of Mathematics*, **59**, 1.
Inönü, E. and Wigner, E. P. (1952). *Nuovo Cimento*, **IX**, 705.
von Neumann, J. (1931). *Mathematische Annalen*, **104**, 570.